Published by Institute of Physics Publishing for SISSA

Received: *November 30, 2005*
Accepted: *January 30, 2006*
Published: *February 14, 2006*

Six-dimensional gauge theory on the chiral square

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Abstract: We construct gauge theories in two extra dimensions compactified on the chiral square, which is a simple compactification that leads to chiral fermions in four dimensions. Stationarity of the action on the boundary specifies the boundary conditions for gauge fields. Any six-dimensional gauge field decomposed in Kaluza-Klein modes includes a tower of heavy spin-1 particles whose longitudinal polarizations are linear combinations of the extra-dimensional components, and a tower of heavy spin-0 particles corresponding to the orthogonal combinations. These linear combinations depend on the Kaluza-Klein numbers, and are independent of the gauge fixing. If the gauge symmetry is broken by the vacuum expectation value of a six-dimensional scalar, at each Kaluza-Klein level three spinless fields in the adjoint representation mix to provide the longitudinal polarization of the spin-1 mode, leaving the orthogonal states as two spin-0 particles. We derive the interactions of the Kaluza-Klein modes for generic gauge theories, laying the groundwork for the Standard Model in two universal extra dimensions, and more generally for future model building and phenomenological studies.

Keywords: Beyond Standard Model, Field Theories in Higher Dimensions.

Contents

1. Introduction

Six-dimensional (6D) gauge theories have intriguing properties that make them potentially relevant for extensions of the standard model of particle physics [1]. For example, the global $SU(2)_W$ gauge anomaly cancels only in the case where the number of quark and lepton generations is a multiple of three [2]. Furthermore, simple compactifications of two dimensions preserve a discrete symmetry which is a subgroup of the 6D Lorentz group, such that the neutrino masses are forced to be of the Dirac type and proton decay is adequately suppressed even when baryon number is maximally violated at the TeV scale [3].

As with any theory that has fermions in more than four dimensions, a major constraint imposed by the observed properties of quarks and leptons is that the compactification allows the existence of chiral fermions in the effective 4D theory obtained by integration over the extra-dimensional coordinates. The simplest compactification of two extra dimensions, namely on a torus, does not satisfy this constraint. A toroidal compactification, which can be viewed as a parallelogram with the opposite sides being identified, leads to 4D fermions that are vector-like with respect to any gauge symmetry. The next simplest compactification, a parallelogram with the adjacent sides being identified, automatically leaves at most a single 4D fermion of definite chirality as the zero mode of any chiral 6D

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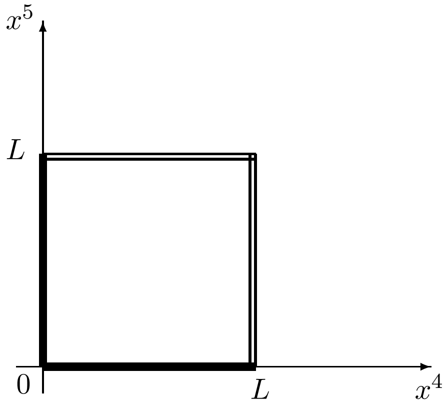

Figure 1: The chiral square: the sides marked by thick lines are identified, and the sides marked by a double line are also identified.

fermion [4]. Identifying adjacent sides requires these to have the same length L, so that the parallelogram has to be a rhombus in this case. For simplicity, we consider the most symmetric compactification of this type: a square. As pointed out in [5], this configuration is naturally preferred by radion and moduli stabilization mechanisms, a simple example of which is the stabilization by quantum, Casimir-like effects. We refer to this compactification as the "chiral square".

In this paper we study 6D gauge theories compactified on the chiral square. There are various questions that we address: what are the boundary conditions for gauge fields on the chiral square? what kind of gauge fixing conditions may be imposed in the 6D theory? what is the spectrum of Kaluza-Klein (KK) modes for gauge fields? how does the Higgs mechanism work if the heavy KK gauge fields acquire masses both from compactification and from spontaneous breaking via a vacuum expectation value (VEV)? what are the interactions of the KK modes? Related studies of gauge theories in six dimensions have appeared in [6 – 8].

Before tackling gauge fields, let us recapitulate some properties of the chiral square derived in ref. [4]. Figure 1 shows a chiral square, with adjacent sides identified. This space has the topology of a sphere, but has a flat metric except for conical singularities at $(0, 0)$ and (L, L) (deficit angle of $3\pi/2$ each) and a third one at the identified points $(0, L) \sim (L, 0)$ (deficit angle of π). From this point of view there is nothing special going on at the sides that are being glued together. It is nevertheless useful to formulate this compactification by considering the above square region on the $x^4 - x^5$ plane, together with boundary conditions on the edges of the square that encode the identification of adjacent sides. In particular, field values should be equal at identified points, modulo possible symmetries of the theory. The physics at identified points is identical if the Lagrangians take the same values for any field configuration:

$$\begin{aligned} \mathcal{L}(x^\mu, y, 0) &= \mathcal{L}(x^\mu, 0, y)\,, \\ \mathcal{L}(x^\mu, y, L) &= \mathcal{L}(x^\mu, L, y)\,, \end{aligned} \tag{1.1}$$

for any $y \in [0, L]$. For a free field Φ, this requirement is consistent with

$$\Phi(x^\mu, y, 0) = e^{i\theta} \Phi(x^\mu, 0, y)\,, \tag{1.2}$$

for an arbitrary phase θ, provided one requires a "smoothness" condition on the derivatives normal to the "edges" of the square

$$\partial_5 \Phi|_{(x^4,x^5)=(y,0)} = -e^{i\theta}\, \partial_4 \Phi|_{(x^4,x^5)=(0,y)}\ . \tag{1.3}$$

Similar relations should be imposed at $(L, y) \sim (y, L)$. However, it was found in [4] that this system admits nontrivial solutions only when θ takes one of the four discrete values $n\pi/2$, for $n = 0, 1, 2, 3$. Only those fields that satisfy boundary conditions corresponding to $n = 0$ admit a zero-mode. Furthermore, when considering 6D Weyl fermions, one finds that their 4D left- and right-handed chiralities obey boundary conditions corresponding to integers that differ by one: $n_L - n_R = \pm 1$, where the sign depends on the 6D chirality. Hence, fermions propagating on this space lead necessarily to a chiral low-energy theory: at most one of the left- or right-handed chiralities has a zero mode. This compactification is equivalent to the T^2/Z_4 orbifold [4].

The chiral square possesses a discrete Z_8 symmetry that acts on the right- and left-handed components of 6D Weyl spinors as

$$\begin{aligned} \Psi_{\pm R}(x^\mu, x^4, x^5) &\mapsto e^{-i(n_R^\pm \pm 1/2)\pi/2}\, \Psi_{\pm R}(x^\mu, x^4, x^5)\,, \\ \Psi_{\pm L}(x^\mu, x^4, x^5) &\mapsto e^{-i(n_L^\pm \mp 1/2)\pi/2}\, \Psi_{\pm L}(x^\mu, x^4, x^5)\,, \end{aligned} \tag{1.4}$$

where $+$ or $-$ label the 6D chirality, and $n_L^\pm$, $n_R^\pm$ label the boundary conditions satisfied by $\Psi_{\pm L}$ and $\Psi_{\pm R}$, respectively. Note that each KK tower transforms as a single entity under the Z_8, i.e. the symmetry commutes with KK number. This Z_8 symmetry is at the heart of the Dirac nature of neutrinos and the suppression of baryon number violation.

A KK-parity can also be naturally imposed on these scenarios and ensures that the lightest KK particle (LKP) is a viable dark matter candidate [9]. This Z_2 symmetry distinguishes among KK modes and acts as

$$\Phi^{(j,k)}(x^\mu) \mapsto (-1)^{j+k}\, \Phi^{(j,k)}(x^\mu)\,, \tag{1.5}$$

where Φ stands for a field of any spin, and j, k are integers labeling the KK level. The KK-parity has a geometrical interpretation as a rotation by π about the center of the chiral square. In particular, KK-parity requires that localized operators at $(0, 0)$ and (L, L) be identical. Localized operators at $(0, L) \sim (L, 0)$ have coefficients that are, in general, unrelated to those on the previous two conical singularities.

In section 2 we give the appropriate boundary conditions for gauge fields propagating on the chiral square. We concentrate on those boundary conditions that preserve a zero mode, i.e. we do not study the breaking of gauge symmetries by boundary conditions. Next we turn to deriving the self-interactions of the KK modes in the mass eigenstate basis for non-Abelian gauge fields (section 3), gauge interactions of fermions (section 4) and scalars (section 5). In section 6 we analyze spontaneously broken 6D gauge symmetries. We also address in detail the gauge fixing procedure, with and without breaking by nonzero VEV's, discuss the associated ghost action and corresponding KK decomposition, isolate the linear combinations of scalars that provide the longitudinal polarizations for massive gauge fields, and identify the additional scalars coming from the extra dimensional components of the gauge fields. We summarize and conclude in section 7.

2. Abelian gauge fields

Let us first study a 6D Abelian gauge field, $A^\alpha(x^\beta)$ with $\alpha, \beta = 0, 1, \ldots 5$, whose propagation in the x^4, x^5 plane is restricted to a square of size L (x^ν, $\nu = 0, 1, 2, 3$ are the Minkowski spacetime coordinates). The action has the usual quadratic form in the gauge field strength, $F^{\alpha\beta}$,

$$S = \int d^4x \int_0^L dx^4 \int_0^L dx^5 \left(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \mathcal{L}_{GF} \right) , \tag{2.1}$$

and includes a gauge fixing term, $\mathcal{L}_{GF}$, which we choose such that the mixings of A_μ, $\mu = 0, 1, 2, 3$, with A_4 and A_5 vanish:

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \Big[\partial_\mu A^\mu - \xi \left(\partial_4 A_4 + \partial_5 A_5 \right) \Big]^2 , \tag{2.2}$$

where ξ is the gauge fixing parameter.

The action may also include localized kinetic terms at the fixed points $(0, 0)$, (L, L) and $(0, L) \sim (L, 0)$. Such terms are generated radiatively, as was explicitly shown in the cases of 5D theories [10] and of 6D theories compactified on the T^2/Z_2 orbifold [11], and are phenomenologically important [12, 13]. In this work we assume that they are sufficiently small that they can be taken into account perturbatively. Therefore, eq. (2.1) is our starting point for the KK decomposition. We defer a detailed study of localized terms for future work [14].

2.1 Boundary conditions

Integrating by parts, the action (2.1) can be written as

$$\begin{aligned} S = \int d^4x \int_0^L dx^4 \int_0^L dx^5 \Big\{ & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + \frac{1}{2} \left[(\partial_4 A_\mu)^2 + (\partial_5 A_\mu)^2 \right] \\ & + \frac{1}{2} \left[(\partial_\mu A_4)^2 + (\partial_\mu A_5)^2 - \xi (\partial_4 A_4 + \partial_5 A_5)^2 - (\partial_4 A_5 - \partial_5 A_4)^2 \right] \\ & + \partial_4 \left[A_4 \partial_\mu A^\mu \right] + \partial_5 \left[A_5 \partial_\mu A^\mu \right] \Big\} . \end{aligned} \tag{2.3}$$

The last two terms are surface terms that are important in determining the possible boundary conditions.

The variation of S with respect to A^α must vanish everywhere in the bulk, leading to the following field equations:

$$\begin{aligned}
\partial^\mu F_{\mu\nu} + \frac{1}{\xi}\partial_\mu\partial_\nu A^\mu &= \left(\partial_4^2 + \partial_5^2\right) A_\nu\,, \\
\left(\partial_\mu\partial^\mu - \xi\partial_4^2 - \partial_5^2\right) A_4 &= (\xi - 1)\partial_4\partial_5 A_5\,, \\
\left(\partial_\mu\partial^\mu - \partial_4^2 - \xi\partial_5^2\right) A_5 &= (\xi - 1)\partial_4\partial_5 A_4\,.
\end{aligned} \tag{2.4}$$

Furthermore, we require the surface terms in δS to vanish everywhere on the boundary:

$$\begin{aligned}
\int d^4x \Bigg\{ \int_0^L dx^4 \, \Big[F_{5\mu}\delta A^\mu + F_{45}\delta A_4 + \left(\partial_\mu A^\mu - \xi\partial_4 A_4 - \xi\partial_5 A_5\right)\delta A_5 \Big]\Big|_{x^5=0}^{x^5=L} \\
+ \int_0^L dx^5 \, \Big[F_{4\mu}\delta A^\mu - F_{45}\delta A_5 + \left(\partial_\mu A^\mu - \xi\partial_4 A_4 - \xi\partial_5 A_5\right)\delta A_4 \Big]\Big|_{x^4=0}^{x^4=L} \Bigg\} = 0\,.
\end{aligned} \tag{2.5}$$

This leads to boundary conditions that guarantee a well-defined, self-adjoint problem, which in turn ensures that the differential operators in eqs. (2.4) possess a complete set of orthogonal eigenfunctions. Possible localized terms in the original action would give additional contributions to eq. (2.5), and thus would correspond to a modification of the boundary conditions. Requiring that the boundary contributions to δS vanish also guarantees that there is no flow of charges, such as energy or momentum, across the boundary, or that any flow is explicitly associated with localized terms that act as sources for the corresponding charges. As already mentioned, we do not consider localized terms in what follows.

As discussed in section 1, we consider the case where the points $(y, 0)$ and $(0, y)$ are identified in the sense that the Lagrangians at these points are equal, and likewise (y, L) and (L, y) are identified, for any $0 \leq y \leq L$. Given that any matter field Φ satisfies the boundary conditions (1.2) and (1.3), and analogous relations at the boundaries $x^4 = L$ and $x^5 = L$, the requirement that the boundary conditions are compatible with the gauge symmetry implies

$$\begin{aligned}
D_\mu\Phi|_{(x^4,x^5)=(y,0)} &= e^{i\theta}\, D_\mu\Phi|_{(x^4,x^5)=(0,y)}\,, \\
D_4\Phi|_{(x^4,x^5)=(y,0)} &= e^{i\theta}\, D_5\Phi|_{(x^4,x^5)=(0,y)}\,, \\
D_5\Phi|_{(x^4,x^5)=(y,0)} &= -e^{i\theta}\, D_4\Phi|_{(x^4,x^5)=(0,y)}\,.
\end{aligned} \tag{2.6}$$

The first and second equations are derived by differentiating eq. (1.2) with respect to x^μ and y, respectively, and then replacing partial derivatives by covariant ones,

$$D_\alpha = \partial_\alpha - ig_6 A_\alpha\,, \tag{2.7}$$

where the 6D gauge coupling, g_6, has mass dimension -1. The last equation in (2.6) is obtained directly from eq. (1.3).

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Eq. (2.6) implies that the boundary conditions are invariant under 6D gauge transformations only if A_μ, A_4 and A_5 satisfy the "folding" identifications

$$\begin{aligned} A_\mu(y,0) &= A_\mu(0,y)\,, \\ A_4(y,0) &= A_5(0,y)\,, \\ A_5(y,0) &= -A_4(0,y)\,, \end{aligned} \tag{2.8}$$

and the same relations between fields at (y,L) and (L,y). These boundary conditions have also been derived in [8].

Alternatively, one can understand the sign in the last equation of (2.8) by recalling that the folding boundary condition (1.2) is closely related to rotations by $\pi/2$ about the origin of the larger square $-L < x^4, x^5 < L$. Under $(x^4,x^5) \mapsto (-x^5,x^4)$, the gauge field satisfies the covariant transformation law $(A_4,A_5) \mapsto (A_5,-A_4)$, and identifying boundary points that differ by such a rotation leads to eq. (2.8).

In the presence of the folding identifications (2.8), eq. (2.5) implies that either the gauge field values are fixed on the boundary ($\delta A_\alpha = 0$), or else

$$\begin{aligned} F_{\mu 5}(y,0) &= -F_{\mu 4}(0,y)\,, \\ F_{45}(y,0) &= F_{45}(0,y)\,, \\ \Big[\,\partial_\mu A^\mu - \xi\,(\partial_4 A_4 + \partial_5 A_5)\Big]_{(x^4,x^5)=(y,0)} &= \Big[\,\partial_\mu A^\mu - \xi\,(\partial_4 A_4 + \partial_5 A_5)\Big]_{(x^4,x^5)=(0,y)}\,. \end{aligned} \tag{2.9}$$

In the latter case, differentiating eqs. (2.8) with respect to y and combining with eq. (2.9) we find some constraints on the x^4 and x^5 derivatives of A_α on adjacent sides of the square. For A_μ, the full set of boundary conditions reads

$$\begin{aligned} A_\mu(y,0) &= A_\mu(0,y)\,, \\ \partial_4 A_\mu|_{(x^4,x^5)=(y,0)} &= \partial_5 A_\mu|_{(x^4,x^5)=(0,y)}\,, \\ \partial_5 A_\mu|_{(x^4,x^5)=(y,0)} &= -\partial_4 A_\mu|_{(x^4,x^5)=(0,y)}\,, \end{aligned} \tag{2.10}$$

and analogous relations at (y,L) and (L,y). The conditions on the A_4 and A_5 components of the gauge field are more conveniently expressed in terms of the fields $A_\pm = A_4 \pm iA_5$:

$$\begin{aligned} A_\pm(y,0) &= \mp i A_\pm(0,y)\,, \\ \partial_4 A_\pm|_{(x^4,x^5)=(y,0)} &= \mp i\,\partial_5 A_\pm|_{(x^4,x^5)=(0,y)}\,, \\ \partial_5 A_\pm|_{(x^4,x^5)=(y,0)} &= \pm i\,\partial_4 A_\pm|_{(x^4,x^5)=(0,y)}\,. \end{aligned} \tag{2.11}$$

2.2 Kaluza-Klein decomposition

Without loss of generality, we may expand the fields A_μ, $A_\pm$ in terms of complete sets of functions satisfying the boundary conditions (2.10) and (2.11). Using the complete sets of

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functions satisfying these boundary conditions found in [4], we may write

$$
\begin{aligned}
A_\mu(x^\nu, x^4, x^5) &= \frac{1}{L}\left[A_\mu^{(0,0)}(x^\nu) + \sum_{j\geq 1}\sum_{k\geq 0} f_0^{(j,k)}(x^4, x^5) A_\mu^{(j,k)}(x^\nu)\right], \\
A_+(x^\nu, x^4, x^5) &= -\frac{1}{L}\sum_{j\geq 1}\sum_{k\geq 0} f_3^{(j,k)}(x^4, x^5) A_+^{(j,k)}(x^\nu), \\
A_-(x^\nu, x^4, x^5) &= \frac{1}{L}\sum_{j\geq 1}\sum_{k\geq 0} f_1^{(j,k)}(x^4, x^5) A_-^{(j,k)}(x^\nu),
\end{aligned} \tag{2.12}
$$

where j and k are integers, $A_\mu^{(j,k)}$ and $A_\pm^{(j,k)}$ are canonically normalized KK modes, and the KK functions f_n with $n = 0, 1, 2, 3$ are:

$$
\begin{aligned}
f_{0,2}^{(j,k)}(x^4, x^5) &= \frac{1}{1+\delta_{j,0}}\left[\cos\left(\frac{jx^4 + kx^5}{R}\right) \pm \cos\left(\frac{kx^4 - jx^5}{R}\right)\right], \\
f_{1,3}^{(j,k)}(x^4, x^5) &= i\sin\left(\frac{jx^4 + kx^5}{R}\right) \mp \sin\left(\frac{kx^4 - jx^5}{R}\right) .
\end{aligned} \tag{2.13}
$$

These satisfy the two-dimensional Klein-Gordon equation,

$$
\left(\partial_4^2 + \partial_5^2 + M_{j,k}^2\right) f_n^{(j,k)}(x^4, x^5) = 0, \tag{2.14}
$$

where

$$
M_{j,k}^2 \equiv \frac{j^2 + k^2}{R^2}, \tag{2.15}
$$

with $R = L/\pi$, and are normalized so that

$$
\frac{1}{L^2}\int_0^L dx^4 \int_0^L dx^5 \left[f_n^{(j,k)}(x^4, x^5)\right]^* f_n^{(j',k')}(x^4, x^5) = \delta_{j,j'}\,\delta_{k,k'} . \tag{2.16}
$$

Note that $f_1^{(j,k)} = -f_3^{(j,k)*}$, so that the explicit minus sign in the expansion of A_+ shown in eq. (2.12) leads to $A_-^{(j,k)} = A_+^{(j,k)*}$. Derivatives along x^4 or x^5 acting on the KK functions satisfy

$$
\partial_\pm f_n^{(j,k)}(x^4, x^5) = i r_{j,\pm k} M_{j,k} f_{n\mp 1}^{(j,k)}(x^4, x^5), \tag{2.17}
$$

where

$$
\partial_\pm = \partial_4 \pm i\partial_5 \tag{2.18}
$$

and $r_{j,k}$ are complex phases that depend only on the KK-numbers:

$$
r_{j,k} \equiv \frac{j + ik}{\sqrt{j^2 + k^2}} . \tag{2.19}
$$

Before inserting the KK expansions (2.12) into the action (2.3), note that

$$
\begin{aligned}
\partial_4 A_5 - \partial_5 A_4 &= \frac{1}{L}\sum_{j\geq 1}\sum_{k\geq 0} M_{j,k}\, A_H^{(j,k)}(x^\nu) f_0^{(j,k)}(x^4, x^5), \\
\partial_4 A_4 + \partial_5 A_5 &= \frac{1}{L}\sum_{j\geq 1}\sum_{k\geq 0} M_{j,k}\, A_G^{(j,k)}(x^\nu) f_0^{(j,k)}(x^4, x^5),
\end{aligned} \tag{2.20}
$$

where we defined at each KK level two real scalar fields, $A_H^{(j,k)}$ and $A_G^{(j,k)}$, by

$$A_\pm^{(j,k)} = r_{j,\pm k}\left(A_H^{(j,k)} \mp i A_G^{(j,k)}\right) \ . \tag{2.21}$$

The explicit factors of $M_{j,k}$ in eqs. (2.20) ensure that the scalars $A_H^{(j,k)}$ and $A_G^{(j,k)}$ are canonically normalized. It is clear that $A_H^{(j,k)}$ correspond to excitations which are invariant under 6D gauge transformations: $A_\alpha \to A_\alpha + \partial_\alpha \chi/g_6$, where χ is a gauge parameter that, like A_μ in eq. (2.12), has an expansion in terms of $f_0^{(j,k)}$. The orthogonal excitations, $A_G^{(j,k)}$, shift under such a gauge transformation and correspond to the Nambu-Goldstone modes eaten by the massive 4D fields, $A_\mu^{(j,k)}$.

After integrating over x^4 and x^5, we find the 4D Lagrangian for the KK modes (one can check that the last two terms in eq. (2.3) give no contribution). The gauge boson of level $(0,0)$ remains massless, while the gauge bosons of level (j,k) with $j \geq 1$ appear as massive vector particles in an R_ξ gauge, with a Lagrangian

$$-\frac{1}{4}F_{\mu\nu}^{(j,k)}F^{(j,k)\mu\nu} + \frac{1}{2}M_{j,k}^2\left(A_\mu^{(j,k)}\right)^2 - \frac{1}{2\xi}\left(\partial^\mu A_\mu^{(j,k)}\right)^2 \ , \tag{2.22}$$

where $F_{\mu\nu}^{(j,k)} = \partial_\mu A_\nu^{(j,k)} - \partial_\nu A_\mu^{(j,k)}$ is the 4D field strength of level (j,k), and for the zero-mode one just sets $M_{0,0} = 0$. At each (j,k) level with $j \geq 1$ one finds that $A_H^{(j,k)}$ and $A_G^{(j,k)}$, as defined in eqs. (2.20) and (2.21), are mass eigenstates in the gauge defined by eq. (2.2), and are described by the following terms in the 4D Lagrangian:

$$\frac{1}{2}\left[\left(\partial_\mu A_H^{(j,k)}\right)^2 - M_{j,k}^2\left(A_H^{(j,k)}\right)^2 + \left(\partial_\mu A_G^{(j,k)}\right)^2 - \xi M_{j,k}^2\left(A_G^{(j,k)}\right)^2\right] \ . \tag{2.23}$$

One can explicitly check that the field equations (2.4), when expressed in terms of $A_H^{(j,k)}$ and $A_G^{(j,k)}$, are satisfied. In the unitary gauge, $\xi \to \infty$, only the $A_H^{(j,k)}(x)$ scalars propagate with masses $M_{j,k}$, whereas the fields $A_G^{(j,k)}(x)$ are the Nambu-Goldstone bosons eaten by the $A_\mu^{(j,k)}(x)$ KK gauge bosons.

3. Non-Abelian gauge fields

The boundary conditions, KK decomposition, and identification of the physical states in the case of non-Abelian gauge fields are analogous to the ones discussed in section 2 for Abelian fields. In this section we present the self-interactions of the KK modes associated with non-Abelian gauge fields, and then we study the ghost fields required by gauge fixing.

3.1 Self-interactions

The kinetic term of a 6D non-Abelian gauge field, A_α^a, where a labels the generators of the adjoint representation, is given by

$$-\frac{1}{4}F_{\alpha\beta}^a F^{a\,\alpha\beta} = -\frac{1}{4}\left(F_{\mu\nu}^a F^{a\,\mu\nu} - 2F_{+\mu}^a F_-^{a\mu}\right) + \frac{1}{8}\left(F_{+-}^a\right)^2 \ . \tag{3.1}$$

The gauge field strengths introduced here are defined by

$$
\begin{aligned}
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_6 f^{abc} A^b_\mu A^c_\nu \,, \\
F^a_{\pm\mu} &= \partial_\pm A^a_\mu - \partial_\mu A^a_\pm + g_6 f^{abc} A^b_\pm A^c_\mu = F^a_{4\mu} \pm i F^a_{5\mu} \,, \\
F^a_{+-} &= \partial_+ A^a_- - \partial_- A^a_+ + g_6 f^{abc} A^b_+ A^c_- = -2i F^a_{45} \,,
\end{aligned}
\tag{3.2}
$$

where f^{abc} are the group structure constants.

The trilinear terms included in eq. (3.1) are given by

$$
-\frac{g_6}{4} f^{abc} \left\{ 4 A^{a\,\mu} A^b_\nu \, \partial_\mu A^{c\,\nu} + \left[A^a_- A^b_+ \, \partial_+ A^c_- + 2 A^a_- A^{b\,\mu} \left(\partial_\mu A^c_+ - \partial_+ A^c_\mu \right) + \text{H.c.} \right] \right\} \,. \tag{3.3}
$$

Upon KK decomposition and integration over the x^4 and x^5 coordinates, these give rise in the 4D Lagrangian to trilinear interactions (proportional to the 4D gauge coupling, $g_4 = g_6/L$) among spin-1 modes,

$$
- g_4 f^{abc} \delta^{(j_1 k_1)(j_2 k_2)(j_3 k_3)}_{0,0,0} A^{(j_1,k_1)\,a}_\mu A^{(j_2,k_2)\,b}_\nu \, \partial^\mu A^{(j_3,k_3)\,c\,\nu} \,, \tag{3.4}
$$

as well as trilinear interactions involving one, two or three scalar gauge fields:

$$
\begin{aligned}
\frac{g_4}{2} f^{abc} \delta^{(j_1 k_1)(j_2 k_2)(j_3 k_3)}_{1,3,0} A^{(j_1,k_1)\,a}_- \Big[& -i r_{j_2,k_2} M_{j_2,k_2} A^{(j_2,k_2)\,b}_\mu A^{(j_3,k_3)\,c\,\mu} - \left(\partial^\mu A^{(j_2,k_2)\,b}_+ \right) A^{(j_3,k_3)\,c}_\mu \\
& + \frac{i}{2} r_{j_3,k_3} M_{j_3,k_3} A^{(j_2,k_2)\,b}_+ A^{(j_3,k_3)\,c}_- \Big] + \text{H.c.}
\end{aligned}
\tag{3.5}
$$

Here we used eq. (2.17) to express the $\partial_\pm$ derivatives in terms of the KK masses.

We have introduced the following notation:

$$
\delta^{(j_1,k_1)\dots(j_r,k_r)}_{n_1,\dots,n_r} \equiv \frac{1}{L^2} \int_0^L dx^4 \int_0^L dx^5 \, f^{(j_1,k_1)}_{n_1} \dots f^{(j_r,k_r)}_{n_r} \,. \tag{3.6}
$$

This integral was computed for $r = 3$ in ref. [4], and the result in the particular cases relevant for trilinear interactions read

$$
\begin{aligned}
\delta^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}_{n,\overline{n},0} &= \frac{1}{2\,(1+\delta_{j_1,0})\,(1+\delta_{j_2,0})\,(1+\delta_{j_3,0})} \Big[7 \delta_{j_1,0} \delta_{j_2,0} \delta_{j_3,0} + \delta_{j_1,j_3-j_2} \delta_{k_1,k_3-k_2} \\
& + i^n (\delta_{j_1,j_3+k_2} \delta_{k_1,k_3-j_2} + \delta_{j_1,k_2-k_3} \delta_{k_1,j_3-j_2}) + (-i)^n (\delta_{j_1,j_3-k_2} \delta_{k_1,j_2+k_3} + \delta_{j_1,k_3-k_2} \delta_{k_1,j_2-j_3}) \\
& + (-1)^n (\delta_{j_1,j_2+j_3} \delta_{k_1,k_2+k_3} + \delta_{j_1,j_2-j_3} \delta_{k_1,k_2-k_3} + \delta_{j_1,j_2-k_3} \delta_{k_1,j_3+k_2} + \delta_{j_1,j_2+k_3} \delta_{k_1,k_2-j_3}) \Big] \,,
\end{aligned}
\tag{3.7}
$$

where $\overline{n} \equiv -n \bmod 4$, and we have used the fact that $j = 0$ implies $k = 0$.

The quartic terms included in eq. (3.1),

$$
-\frac{g_6^2}{4} f^{abc} f^{ade} \left(A^b_\mu A^c_\nu A^{d\,\mu} A^{e\,\nu} - 2 A^b_\mu A^c_+ A^{d\,\mu} A^e_- - \frac{1}{2} A^b_+ A^c_- A^d_+ A^e_- \right) \,, \tag{3.8}
$$

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lead to the following quartic interactions of the KK modes:

$$
\begin{aligned}
-\frac{g_4^2}{4} f^{abc} f^{ade} \Big[& \delta_{0,0,0,0}^{(j_1,k_1)\cdots(j_4,k_4)} A_\mu^{(j_1,k_1)\,b} A_\nu^{(j_2,k_2)\,c} A^{(j_3,k_3)\,d\,\mu} A^{(j_4,k_4)\,e\,\nu} \\
& + 2\delta_{3,1,0,0}^{(j_1,k_1)\cdots(j_4,k_4)} A_+^{(j_1,k_1)\,c} A_-^{(j_2,k_2)\,e} A_\mu^{(j_3,k_3)\,b} A^{(j_4,k_4)\,d\,\mu} \\
& - \frac{1}{2}\delta_{3,1,3,1}^{(j_1,k_1)\cdots(j_4,k_4)} A_+^{(j_1,k_1)\,b} A_-^{(j_2,k_2)\,c} A_+^{(j_3,k_3)\,d} A_-^{(j_4,k_4)\,e} \Big] \ .
\end{aligned} \tag{3.9}
$$

Of particular interest for phenomenolgy are vertices involving at least a zero mode, and for those one can use eq. (3.7) because

$$\delta_{n,\overline{n},0,0}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)(0,0)} = \delta_{n,\overline{n},0}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)} \ . \tag{3.10}$$

In order to extract the interactions of the physical states, the $A_\pm^{(j,k)\,a}$ complex scalars have to be replaced in eqs. (3.5) and (3.9) by the two real scalar fields, $A_H^{(j,k)\,a}$ and $A_G^{(j,k)\,a}$, as prescribed by eq. (2.21). These heavy 4D scalars are in the adjoint representation of the non-Abelian gauge group, so we will refer to them as “spinless adjoints”.

3.2 Ghost fields

For completeness we now turn to determining the ghost action associated with the gauge fixing term given by the obvious adaptation of eq. (2.2) to non-Abelian fields. This arises from inserting in the path integral a factor

$$\int \mathcal{D}\chi \delta[G(A)] \det\left(\frac{\delta G(A^\chi)}{\delta\chi}\right) = 1 \, , \tag{3.11}$$

where χ is the gauge transformation parameter, A^χ is the transformed gauge field, and the gauge fixing condition is

$$G(A^a) = \partial^\mu A_\mu^a - \xi\left(\partial_4 A_4^a + \partial_5 A_5^a\right) - \omega^a \, , \tag{3.12}$$

for arbitrary functions ω^a. After integrating with a Gaussian weight over ω^a one recovers eq. (2.2). Since

$$(A_\alpha^a)^\chi = A_\alpha^a + \frac{1}{g_6} D_\alpha \chi^a \, , \tag{3.13}$$

with D_α the covariant derivative in the adjoint representation, we find

$$\frac{\delta G(A^\chi)}{\delta\chi} = \frac{1}{g_6}\left[\partial_\mu D^\mu - \xi\left(\partial_4 D_4 + \partial_5 D_5\right)\right] \ . \tag{3.14}$$

These terms in the Lagrangian may be taken into account by including a ghost term in the 6D action, given by

$$-\bar{c}^a \left[\partial_\mu D^\mu - \xi\left(\partial_4 D_4 + \partial_5 D_5\right)\right] c^a \, , \tag{3.15}$$

where c^a is an anti-commuting 6D scalar in the adjoint representation of the gauge group.

The above procedure did not take into account the compactification of the two extra dimensions and the associated boundary conditions. The free part of the 6D ghost Lagrangian (3.15) is

$$-\bar{c}^a \left[\partial_\mu \partial^\mu - \xi\left(\partial_4^2 + \partial_5^2\right)\right] c^a \, , \tag{3.16}$$

which up to the factor ξ is the same as for a scalar. It follows that the KK expansion for the ghost fields can be written as

$$c^a(x^\mu, x^4, x^5) = \frac{1}{L} \sum_{(j,k)} c^{(j,k)\,a}(x^\mu) f_{n_c}^{(j,k)}(x^4, x^5)\,, \tag{3.17}$$

where the $f_{n_c}^{(j,k)}$ belong to one of the sets of KK functions defined in eqs. (2.13). The KK modes $c^{(j,k)\,a}(x^\mu)$ have mass $\sqrt{\xi} M_{j,k}$. It would appear that n_c can take any of the integral values $0, 1, 2, 3$. However, for $n_c = 1, 2, 3$, the ghost fields lack the zero modes necessary to ensure the gauge invariance of amplitudes involving the zero-mode gauge fields. More generally, since $A_\mu(x^\mu, x^4, x^5)$ satisfies boundary conditions with $n = 0$, the necessary relations between the couplings involving only gauge fields and those involving ghosts and gauge fields can be satisfied only if the ghosts satisfy the same boundary conditions as A_μ. Therefore, only the value $n_c = 0$ is allowed. The zero mode $c^{(0,0)}$ is the ghost required by 4D gauge invariance.

After integrating by parts eq. (3.15), the interactions between gauge bosons and ghost fields are given by

$$g_6 f^{abc} (\partial^\mu \bar{c}^a) A_\mu^b c^c - \frac{1}{2}\xi\, g_6 f^{abc} \left[(\partial_+ \bar{c}^a) A_-^b + (\partial_- \bar{c}^a) A_+^b \right] c^c\,, \tag{3.18}$$

where $A_\pm$ were defined before eq. (2.11) and $\partial_\pm$ were defined in (2.18). It is worth pointing out that the boundary conditions for c^a given in eq. (3.17), together with the boundary conditions for A_4, A_5 discussed in section 2, ensure that one can freely integrate by parts eq. (3.18) without generating surface terms.

Using the KK expansions (3.17), as well as eq. (2.20), eq. (3.18) leads to KK interactions between ghosts and gauge fields:

$$g_4 f^{abc} \delta_{0,0,0}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)} \left(\partial^\mu \bar{c}^{(j_1,k_1)\,a} \right) A_\mu^{(j_2,k_2)\,b} c^{(j_3,k_3)\,c}\,, \tag{3.19}$$

and between ghosts and the "spinless adjoints":

$$-\frac{i}{2} g_4 f^{abc} \xi r_{j_1,k_1} M_{j_1,k_1} \delta_{3,1,0}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)} \bar{c}^{(j_1,k_1)\,a} A_-^{(j_2,k_2)\,b} c^{(j_3,k_3)\,c} + \text{H.c.}\,, \tag{3.20}$$

where we used eq. (2.17) to simplify the derivative, and $A_\pm^{(j,k)}$ are given in terms of the mass eigenstates $A_H^{(j,k)}$ and $A_G^{(j,k)}$ by eq. (2.21).

4. Gauge interactions of fermions

The 6D chiral fermions have four degrees of freedom, and the Dirac equation can be written using a set of six anti-commuting matrices, Γ^α. We use here the following 8×8 representation:

$$\Gamma^\mu = \gamma^\mu \otimes \sigma^0 \quad , \quad \Gamma^{4,5} = i\gamma_5 \otimes \sigma^{1,2} \tag{4.1}$$

where γ^μ, are the 4D γ-matrices, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, σ^0 is the 2×2 unit matrix and σ^i are the Pauli matrices. The 6D fermions have $+$ or $-$ chirality, defined by the eigenvalue of the $\overline{\Gamma} = -\gamma_5 \otimes \sigma^3$ chirality operator: $\overline{\Gamma}\Psi_\pm = \pm\Psi_\pm$. A 6D chiral fermion includes both 4D chiralities: $\gamma_5 \otimes \sigma^0 \Psi_{\pm L} = -\Psi_{\pm L}$ and $\gamma_5 \otimes \sigma^0 \Psi_{\pm R} = \Psi_{\pm R}$.

It is useful to define

$$\Gamma^\pm = \frac{1}{2}\left(\Gamma^4 \pm i\Gamma^5\right) \, , \tag{4.2}$$

and to observe that

$$\begin{aligned} \Gamma^4 P_L P_\pm &= \mp i \Gamma^5 P_L P_\pm \, , \\ \Gamma^4 P_R P_\pm &= \pm i \Gamma^5 P_R P_\pm \, , \end{aligned} \tag{4.3}$$

where $P_{L,R}$, $P_\pm$ project on the respective 4D and 6D chiralities, from which it follows that

$$\Gamma^+ \Psi_{+L} = \Gamma^- \Psi_{+R} = \Gamma^- \Psi_{-L} = \Gamma^+ \Psi_{-R} = 0 \, . \tag{4.4}$$

The fermion kinetic term can then be written in terms of 4D chiral components as follows:

$$i\overline{\Psi}_\pm \Gamma^\alpha D_\alpha \Psi_\pm = i\left(\overline{\Psi}_{\pm L}\Gamma^\mu D_\mu \Psi_{\pm L} + \overline{\Psi}_{\pm R}\Gamma^\mu D_\mu \Psi_{\pm R} + \overline{\Psi}_{\pm L}\Gamma^\pm D_\mp \Psi_{\pm R} + \overline{\Psi}_{\pm R}\Gamma^\mp D_\pm \Psi_{\pm L}\right) \, . \tag{4.5}$$

Here $D_\pm = D_4 \pm i D_5$ and D_α is the covariant derivative defined in eq. (2.7) with either A_α being an Abelian gauge field, or $A_\alpha = T^a A^a_\alpha$ where A^a_α is a non-Abelian gauge field and T^a are the gauge group generators corresponding to the representation of $\Psi_\pm$.

The KK expansions of a 6D fermion of $+$ chirality with a left-handed zero mode component are [4],

$$\begin{aligned} \Psi_{+L} &= \frac{1}{L}\left[\Psi^{(0,0)}_{+L}(x^\nu) + \sum_{j\geq 1}\sum_{k\geq 0} f^{(j,k)}_0(x^4,x^5)\Psi^{(j,k)}_{+L}(x^\nu)\right] \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, , \\ \Psi_{+R} &= -\frac{i}{L}\sum_{j\geq 1}\sum_{k\geq 0} r_{j,k}\, f^{(j,k)}_3(x^4,x^5)\Psi^{(j,k)}_{+R}(x^\nu) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \, , \end{aligned} \tag{4.6}$$

while those containing a right-handed zero mode are,

$$\begin{aligned} \Psi_{+L} &= \frac{i}{L}\sum_{j\geq 1}\sum_{k\geq 0} r^*_{j,k}\, f^{(j,k)}_1(x^4,x^5)\Psi^{(j,k)}_{+L}(x^\nu) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, , \\ \Psi_{+R} &= \frac{1}{L}\left[\Psi^{(0,0)}_{+R}(x^\nu) + \sum_{j\geq 1}\sum_{k\geq 0} f^{(j,k)}_0(x^4,x^5)\Psi^{(j,k)}_{+R}(x^\nu)\right] \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \, . \end{aligned} \tag{4.7}$$

The phases of the KK functions for left- and right-handed fermions are correlated, but one can choose an overall phase such that the functions $f_n(x^4, x^5)$ with $n = 0, 1, 3$ are given in eq. (2.13). For a fermion of $-$ chirality, the same equations are valid except for an interchange of the L and R indices.

After integrating the 6D fermion kinetic term shown in eq. (4.5) over x^4 and x^5, the 4D Lagrangian includes the usual kinetic terms for all KK modes, mass terms of the type $M_{j,k}\overline{\Psi}^{(j,k)}_{\pm L}\Psi^{(j,k)}_{\pm R}$ + H.c., and interactions among KK modes. The latter ones, in the case

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of a fermion with + chirality having a left-handed zero mode, include interactions of the $\Psi_{+L}^{(j,k)}$ fermions with a spin-1 KK mode,

$$g_4\, \delta_{0,0,0}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}\, \overline{\Psi}_{+L}^{(j_1,k_1)} A_\mu^{(j_2,k_2)} \gamma^\mu \Psi_{+L}^{(j_3,k_3)}\,, \tag{4.8}$$

interactions of the $\Psi_{+R}^{(j,k)}$ fermions with a spin-1 KK mode,

$$-\, g_4\, \delta_{1,0,3}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)} r_{j_1,k_1}^{*} r_{j_3,k_3} \overline{\Psi}_{+R}^{(j_1,k_1)} A_\mu^{(j_2,k_2)} \gamma^\mu \Psi_{+R}^{(j_3,k_3)}\,, \tag{4.9}$$

and gauge interactions of the fermions with the spinless adjoints,

$$-\, i g_4\, \delta_{0,1,3}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)} r_{j_2,k_2}^{*} r_{j_3,k_3}\, \overline{\Psi}_{+L}^{(j_1,k_1)} \left(A_H^{(j_2,k_2)} + i A_G^{(j_2,k_2)}\right) \Psi_{+R}^{(j_3,k_3)} + \text{H.c.} \tag{4.10}$$

The $\delta_{n,\overline{n},0}^{(j_1 k_1)(j_2,k_2)(j_3 k_3)}$ coefficients with $n+\overline{n}=0$ are given in eq. (3.7), the complex numbers $r_{j,k}$ are given in eq. (2.19), and $g_4 = g_6/L$ is the 4D gauge coupling.

In the case of a fermion with + chirality having a right-handed zero mode, the spin-1 KK modes have interactions with $\Psi_{+R}^{(j,k)}$ given by

$$g_4\, \delta_{0,0,0}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}\, \overline{\Psi}_{+R}^{(j_1,k_1)} A_\mu^{(j_2,k_2)} \gamma^\mu \Psi_{+R}^{(j_3,k_3)}\,, \tag{4.11}$$

and interactions with $\Psi_{+L}^{(j,k)}$ given by

$$-\, g_4 \delta_{3,0,1}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)} r_{j_1,k_1} r_{j_3,k_3}^{*} \overline{\Psi}_{+L}^{(j_1,k_1)} A_\mu^{(j_2,k_2)} \gamma^\mu \Psi_{+L}^{(j_3,k_3)}\,, \tag{4.12}$$

while the gauge interactions of the spinless adjoints with the fermions are given by

$$-\, i g_4\, \delta_{0,3,1}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)} r_{j_2,k_2} r_{j_3,k_3}^{*}\, \overline{\Psi}_{+R}^{(j_1,k_1)} \left(A_H^{(j_2,k_2)} - i A_G^{(j_2,k_2)}\right) \Psi_{+L}^{(j_3,k_3)} + \text{H.c.} \tag{4.13}$$

A 6D fermion of − chirality has the same gauge interactions as Ψ_+ except for an interchange of the 4D chiralities. More explicitly, if Ψ_{-R} has a zero mode, then the gauge interactions of Ψ_{-R} and Ψ_{-L} are as in eqs. (4.8)–(4.10) with an interchange of the L and R indices. If Ψ_{-L} has a zero mode, then Ψ_{-R} and Ψ_{-L} have gauge interactions given by eqs. (4.11)–(4.13) with L and R interchanged.

5. Gauge interactions of scalars

Consider now a 6D scalar field Φ transforming under a certain nontrivial representation of a gauge symmetry, with an action given by

$$S_\Phi = \int d^4x \int_0^L dx^4 \int_0^L dx^5 \left[(D_\alpha \Phi)^\dagger D^\alpha \Phi - M_\Phi^2 \Phi^\dagger \Phi - \frac{\lambda_6}{2} \left(\Phi^\dagger \Phi\right)^2 \right]\,, \tag{5.1}$$

where λ_6 is a parameter of mass dimension −2, and D_α is the covariant derivative associated with a gauge field A_α^a, as in eq. (2.7). As in the previous section, we use the short-hand notation $A_\alpha = T^a A_\alpha^a$ where T^a are the gauge group generators corresponding to the representation of Φ.

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The KK decomposition of the scalar has been derived in [4]:

$$\Phi(x^\mu, x^4, x^5) = \frac{1}{L} \sum_{(j,k)} \Phi^{(j,k)}(x^\mu) f_n^{(j,k)}(x^4, x^5)\,, \tag{5.2}$$

with $n = 0, 1, 2,$ or 3. The scalar KK modes $\Phi^{(j,k)}$ have masses

$$M_\Phi^{(j,k)} = \sqrt{M_\Phi^2 + M_{j,k}^2}\,, \tag{5.3}$$

where $M_{j,k}$ are the KK masses given in eq. (2.15).

Using the KK decomposition of gauge fields given in eq. (2.12), and integrating over the extra dimensional coordinates, we find that the 4D Lagrangian includes interactions of two KK scalars with one spin-1 KK field,

$$i g_4 \delta^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}_{\overline{n},0,n} \Phi^{(j_1,k_1)\dagger} A_\mu^{(j_2,k_2)} \partial^\mu \Phi^{(j_3,k_3)} + \text{H.c.}\,, \tag{5.4}$$

as well as interactions of two KK scalars with two spin-1 KK fields,

$$g_4^2 \delta^{(j_1,k_1)\ldots(j_4,k_4)}_{\overline{n},0,0,n} \Phi^{(j_1,k_1)\dagger} A_\mu^{(j_2,k_2)} A^{(j_3,k_3)\,\mu}\, \Phi^{(j_4,k_4)}\ . \tag{5.5}$$

In particular, the interactions of the $A_\mu^{(0,0)}$ fields are dictated by 4D gauge invariance.

The 4D Lagrangian also includes interactions of two $\Phi^{(j,k)}$ scalars with one of the $A_H^{(j,k)}$ and $A_G^{(j,k)}$ spinless adjoints,

$$\begin{aligned}\frac{g_4}{2} M_{j_3,k_3} \Phi^{(j_1,k_1)\dagger} \Bigg[\Big(\delta^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}_{\overline{n},1,n-1} r^*_{j_2,k_2} r_{j_3,k_3} - \delta^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}_{\overline{n},3,n+1} r_{j_2,k_2} r^*_{j_3,k_3} \Big) A_H^{(j_2,k_2)} \\ + i \Big(\delta^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}_{\overline{n},3,n+1} r_{j_2,k_2} r^*_{j_3,k_3} + \delta^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}_{\overline{n},1,n-1} r^*_{j_2,k_2} r_{j_3,k_3} \Big) A_G^{(j_2,k_2)} \Bigg] \Phi^{(j_3,k_3)} + \text{H.c.}\,,\end{aligned} \tag{5.6}$$

[here we replaced the derivatives using eq. (2.17)], interactions of two $\Phi^{(j,k)}$ scalars with two of the spinless adjoints,

$$g_4^2 r_{j_2,k_2} r^*_{j_3,k_3} \delta^{(j_1,k_1)\cdots(j_4,k_4)}_{\overline{n},3,1,n}\, \Phi^{(j_1,k_1)\dagger} \left(A_H^{(j_2,k_2)} - i A_G^{(j_2,k_2)} \right) \left(A_H^{(j_3,k_3)} + i A_G^{(j_3,k_3)} \right) \Phi^{(j_4,k_4)}\,, \tag{5.7}$$

and, finally, self-interactions of the $\Phi^{(j,k)}$ scalars

$$-\frac{\lambda_4}{2} \delta^{(j_1,k_1)\ldots(j_4,k_4)}_{\overline{n},n,\overline{n},n}\, \Phi^{(j_1,k_1)\dagger} \Phi^{(j_2,k_2)} \Phi^{(j_3,k_3)\dagger} \Phi^{(j_4,k_4)}\,, \tag{5.8}$$

where $\lambda_4 = \lambda_6 / L^2$ is the 4D quartic coupling.

6. Spontaneous symmetry breaking

We now discuss the case where the gauge symmetry is broken by the VEV of a 6D scalar Φ. The action is given in eq. (5.1), with $M_\Phi^2 < 0$. By adding an irrelevant constant, the terms in the 6D Lagrangian that include Φ can be rewritten as

$$\mathcal{L}_\Phi = |D_\alpha \Phi|^2 - \frac{\lambda_6}{2} \left(\Phi^\dagger \Phi - \frac{1}{2} v_6^2 \right)^2\,, \tag{6.1}$$

where $v_6 > 0$ is the 6D VEV, which has mass dimension two and is defined by the relation

$$-M_\Phi^2 = \frac{1}{2}\lambda_6 v_6^2 \, . \tag{6.2}$$

Let us focus on the case where the gauge group is SU(N) and the scalar Φ is in the fundamental representation of the gauge group. The formulae we derive below also apply (with trivial modifications) to SO(N) or U(1) gauge groups. In order to analyze the spectrum and interactions in the presence of the 6D VEV, we parameterize Φ as

$$\Phi = \frac{1}{\sqrt{2}}(v_6 + h)\, U_\eta \hat{\Phi}_0 \, , \tag{6.3}$$

where $\hat{\Phi}_0 = (0, \ldots, 0, 1)$ defines the direction of the VEV, h is the single 6D real scalar that is orthogonal to the Nambu-Goldstone modes, and U_η is an unitary matrix that depends on the 6D Nambu-Goldstone bosons, η^a:

$$U_\eta = e^{i\eta/v_6} \, , \tag{6.4}$$

where $\eta = \eta^a X^a$, with X^a the broken generators. The sum over a includes the $2N-1$ generators of the coset $\mathrm{SU}(N)/\,\mathrm{SU}(N-1)$. In terms of h and η, the Lagrangian (6.1) is given by

$$\mathcal{L}_\Phi = \frac{1}{2}(\partial_\alpha h)^2 - \frac{\lambda_6}{8}\, h^2\,(2v_6 + h)^2 + \frac{1}{2}(v_6 + h)^2\, \hat{\Phi}_0^\dagger \left| U_\eta^\dagger \partial_\alpha U_\eta - i g_6 U_\eta^\dagger A_\alpha U_\eta \right|^2 \hat{\Phi}_0 \, . \tag{6.5}$$

where $A_\alpha = A_\alpha^a T^a$, with a running over all group generators T^a.

6.1 Physical states

Expanding U_η in powers of η, we find the terms in $\mathcal{L}_\Phi$ that are quadratic in η or A_α:

$$\frac{1}{2} v_6^2\, \hat{\Phi}_0^\dagger \left(\frac{1}{v_6}\partial_\alpha \eta - g_6 A_\alpha \right)^2 \hat{\Phi}_0 \, . \tag{6.6}$$

After integration by parts, these become

$$\frac{1}{2}\hat{\Phi}_0^\dagger \left[(\partial_\alpha \eta)^2 + g_6 v_6 \left\{ \eta, \partial_\alpha A^\alpha \right\} + g_6^2 v_6^2 A_\alpha A^\alpha \right] \hat{\Phi}_0 \, , \tag{6.7}$$

where $\{\ldots,\ldots\}$ is the anticommutator. We see that η^a mixes with the broken components of both A_μ^a and $\partial_4 A_4^a + \partial_5 A_5^a$, where the latter includes the KK modes $A_G^{(j,k)\,a}$ that are eaten by the spin-1 modes in the limit of zero VEV [see eq. (2.20)]. In order to make the physics more transparent, we modify the gauge fixing term (2.2) to

$$\mathcal{L}_{GF}^v = -\frac{1}{2\xi}\left[\, \partial_\mu A^{\mu\, a} - \xi\left(\partial_4 A_4^a + \partial_5 A_5^a \right) + \xi g_6 v_6 \eta^a \right]^2 \, , \tag{6.8}$$

where it is understood that η^a is non-vanishing only along the direction of the broken generators. The terms involving the unbroken components of A_μ^a remain as in eq. (2.2). Since $\hat{\Phi}_0^\dagger \left\{ \eta, \partial_\alpha A^\alpha \right\} \hat{\Phi}_0 = 2\eta^a \partial_\alpha A^{\alpha\, a}$, where a runs over the broken generators, the crossed terms in eq. (6.8) involving η^a and A^a_μ cancel the corresponding terms in (6.7). The remaining terms in eqs. (6.7) and (6.8), involving η, A_4 and A_5, are then

$$\frac{1}{2}\hat{\Phi}_0^\dagger \left[(\partial_\alpha \eta)^2 - g_6^2 v_6^2 A_+ A_- - \xi\, g_6^2 v_6^2 \eta^2 + g_6 v_6 (\xi - 1)\{\eta, \partial_4 A_4 + \partial_5 A_5\}\right] \hat{\Phi}_0 \,, \tag{6.9}$$

together with the second line in eq. (2.3). Here we used $A^a_\alpha A^{\alpha\, a} = A^a_\mu A^{\mu\, a} - A^a_+ A^a_-$, where $A^a_\pm = A_4 \pm i A_5$.

We turn next to the KK decomposition. This is achieved by using the KK expansions for A^a_μ and $A^a_\pm$, eqs. (2.12), supplemented by

$$\begin{aligned} h(x^\mu, x^4, x^5) &= \frac{1}{L}\sum_{(j,k)} h^{(j,k)}(x^\mu) f_0^{(j,k)}(x^4, x^5)\,, \\ \eta^a(x^\mu, x^4, x^5) &= \frac{1}{L}\sum_{(j,k)} \eta^{(j,k)\, a}(x^\mu) f_0^{(j,k)}(x^4, x^5)\,. \end{aligned} \tag{6.10}$$

In assuming a constant VEV for Φ we have implicitly imposed that it satisfy boundary conditions with $n = 0$, a fact we used in eqs. (6.10).

Using eqs. (2.20) as well as the orthogonality of the KK functions $f_n^{(j,k)}$, it is now straightforward to obtain the new terms in the KK action. One finds new mass contributions proportional to the VEV for the broken components of $A_\mu^{(j,k)\, a}$ and $A_H^{(j,k)\, a}$ modes, where the latter ones are defined in eq. (2.21):

$$-\frac{1}{2} g_4^2 v_4^2\, \hat{\Phi}_0^\dagger \left(A_\mu^{(j,k)} A^{(j,k)\, \mu} + A_H^{(j,k)} A_H^{(j,k)} \right) \hat{\Phi}_0\,. \tag{6.11}$$

These, together with the mass contributions due to momentum in extra dimensions, shown in eqs. (2.22) and (2.23), lead to masses for these modes given by

$$M_A^{(j,k)} = \sqrt{M_{j,k}^2 + g_4^2 v_4^2}\,, \tag{6.12}$$

with $M_{j,k}$ defined by eq. (2.15). We chose to express these results in terms of the 4D gauge coupling, $g_4 = g_6/L$, and the 4D VEV, $v_4 = v_6 L$, with mass dimension one. Also, from eq. (6.5), the masses of the $h^{(j,k)}$ real scalars are given by

$$M_h^{(j,k)} = \sqrt{M_{j,k}^2 + \lambda_4 v_4^2}\,, \tag{6.13}$$

where $\lambda_4 = \lambda_6/L^2$ is the 4D quartic coupling.

Finally, the mass terms involving $\eta^{(j,k)}$ and $A_G^{(j,k)}$ [see eqs. (6.9) and (2.23)] mix these modes, such that, for the components along the broken generators, the physical states are given by the linear combinations

$$\begin{aligned} \tilde{A}_G^{(j,k)\, a} &= \frac{1}{M_A^{(j,k)}} \left(M_{j,k} A_G^{(j,k)\, a} + g_4 v_4\, \eta^{(j,k)\, a} \right)\,, \\ \tilde{\eta}^{(j,k)\, a} &= \frac{1}{M_A^{(j,k)}} \left(M_{j,k} \eta^{(j,k)\, a} - g_4 v_4 A_G^{(j,k)\, a} \right)\,, \end{aligned} \tag{6.14}$$

for $j > 0$ and $k \geq 0$, while for $j = k = 0$ we define

$$\tilde{A}_G^{(0,0)\,a} = \eta^{(0,0)\,a} \ . \tag{6.15}$$

This latter mode is eaten by the would-be zero-mode of the gauge KK tower. The free Lagrangian terms for these states are

$$\frac{1}{2}\hat{\Phi}_0^\dagger \left[\left(\partial_\mu \tilde{\eta}^{(j,k)}\right)^2 - \left(M_A^{(j,k)}\tilde{\eta}^{(j,k)}\right)^2 + \left(\partial_\mu \tilde{A}_G^{(j,k)}\right)^2 - \xi M_{(j,k)}^2 \left(\tilde{A}_G^{(j,k)}\right)^2\right]\hat{\Phi}_0 \,, \tag{6.16}$$

while for the unbroken components it is given by

$$\frac{1}{2}\left[\left(\partial_\mu A_G^{(j,k)\,a}\right)^2 - \xi M_{j,k}^2 \left(A_G^{(j,k)\,a}\right)^2\right] \ . \tag{6.17}$$

Note that under an infinitesimal 6D gauge transformation

$$A_\alpha^a \mapsto A_\alpha^a + \frac{1}{g_6}\partial_\alpha \chi^a + \cdots \,, \qquad \eta^a \mapsto \eta^a + v_6 \chi^a + \cdots \,, \tag{6.18}$$

one has $\partial_4 A_4^a + \partial_5 A_5^a \mapsto \partial_4 A_4^a + \partial_5 A_5^a + (\partial_4^2 + \partial_5^2)\chi^a/g_6 + \cdots$, which after KK expansion, eq. (2.20), translates at the linear level into

$$A_G^{(j,k)} \mapsto A_G^{(j,k)} + \frac{1}{g_4} M_{j,k}\chi^{(j,k)} + \cdots \,, \qquad \eta^{(j,k)} \mapsto \eta^{(j,k)} + v_4 \chi^{(j,k)} + \cdots \ . \tag{6.19}$$

In the above, the dimensionless gauge transformation parameter, χ, has an expansion

$$\chi(x^\mu, x^4, x^5) = \sum_{(j,k)} \chi^{(j,k)}(x^\mu) f_0^{(j,k)}(x^4, x^5) \,, \tag{6.20}$$

and the $\ldots$ stand for higher order terms that, in general, mix the KK levels. It is then clear that the inhomogeneus piece of the gauge transformation drops from $\tilde{\eta}^{(j,k)\,a}$, while the orthogonal combination, $\tilde{A}_G^{(j,k)\,a}$, shifts under the gauge transformation. $\tilde{A}_G^{(j,k)}$ is the generalization of the would-be Nambu-Goldstone boson of sections 2 and 3 to the case where the gauge symmetry is broken by a constant scalar VEV. Note that in the unitary gauge, $\xi \to \infty$, each KK level gauge boson eats a linear combination of A_4, A_5 and η.

As a result of the modification arising from v_6 in the gauge fixing term, eq. (6.8), the ghost Lagrangian, eq. (3.15), receives a new contribution

$$-\xi g_6^2 v_6^2 \, \bar{c}^a c^a \,, \tag{6.21}$$

where a runs over the broken components only. Hence, in the presence of the Higgs VEV, the KK masses associated with the broken components of the ghost fields are $\sqrt{\xi}M_A^{(j,k)}$, where $M_A^{(j,k)}$ was defined in eq. (6.12), while those associated with the unbroken ones remain at $\sqrt{\xi}M_{j,k}$.

6.2 Trilinear and quartic couplings

The interactions among KK modes follow from the nonlinear terms in eq. (6.5). These can be derived by expanding U_η in a power series:

$$
\begin{aligned}
U_\eta^\dagger \partial_\alpha U_\eta &= i\partial_\alpha \hat{\eta} - \frac{1}{2}\left[\partial_\alpha \hat{\eta}, \hat{\eta}\right] - \frac{i}{3!}\left[\left[\partial_\alpha \hat{\eta}, \hat{\eta}\right], \hat{\eta}\right] + \cdots\,, \\
U_\eta^\dagger A_\alpha U_\eta &= A_\alpha + i\left[A_\alpha, \hat{\eta}\right] - \frac{1}{2}\left[\left[A_\alpha, \hat{\eta}\right], \hat{\eta}\right] - \frac{i}{3!}\left[\left[\left[A_\alpha, \hat{\eta}\right], \hat{\eta}\right], \hat{\eta}\right] + \cdots\,,
\end{aligned} \tag{6.22}
$$

where $[\ldots,\ldots]$ is the commutator and $\hat{\eta} = \eta/v_6$.

The interaction terms up to quadratic order in η and A_α that appear in the Lagrangian $\mathcal{L}_\Phi$ are

$$
-\frac{\lambda_6}{8}\left(4v_6 h^3 + h^4\right) + \frac{1}{2}\left(2v_6 h + h^2\right)\,\hat{\Phi}_0^\dagger \left(\partial_\alpha \hat{\eta} - g_6 A_\alpha\right)^2 \hat{\Phi}_0\,. \tag{6.23}
$$

Higher order terms in η and A_α include additional trilinear and quartic interactions involving the h scalar,

$$
\frac{i}{2} v_6\left(v_6 + 2h\right)\,\hat{\Phi}_0^\dagger \left\{\frac{1}{2}\left\{\partial_\alpha \hat{\eta}, \left[\partial^\alpha \hat{\eta}, \hat{\eta}\right]\right\} + g_6\left[\hat{\eta}\left(\partial_\alpha \hat{\eta}\right) A^\alpha - \left(\partial_\alpha \hat{\eta}\right) A^\alpha \hat{\eta}\right]\right\}\hat{\Phi}_0\,, \tag{6.24}
$$

interactions of three η fields with an A_α,

$$
\frac{g_6}{2} v_6^2\,\hat{\Phi}_0^\dagger \Big[\left(\partial_\alpha \hat{\eta}\right)\left[A^\alpha, \hat{\eta}\right]\hat{\eta} - \hat{\eta}\left[A^\alpha, \hat{\eta}\right]\partial_\alpha \hat{\eta} + \hat{\eta}\left(\partial_\alpha \hat{\eta}\right)\left[\hat{\eta}, A^\alpha\right] - \left[\hat{\eta}, A^\alpha\right]\left(\partial_\alpha \hat{\eta}\right)\hat{\eta}\Big]\,\hat{\Phi}_0\,, \tag{6.25}
$$

quartic interactions among the η fields,

$$
-\frac{1}{2} v_6^2\,\hat{\Phi}_0^\dagger \left\{\frac{1}{3!}\Big(\left(\partial_\alpha \hat{\eta}\right)\left[\left[\partial_\alpha \hat{\eta}, \hat{\eta}\right], \hat{\eta}\right] + \text{H.c.}\Big) + \frac{1}{4}\left[\partial_\alpha \hat{\eta}, \hat{\eta}\right]^2\right\}\hat{\Phi}_0\,, \tag{6.26}
$$

and interactions involving five or more fields.

Replacing the KK expansions, eq. (6.23) leads to trilinear interactions involving the spin-1 modes

$$
v_4\,\delta_{0,0,0}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)} h^{(j_1,k_1)}\hat{\Phi}_0^\dagger \left(\frac{1}{v_4}\partial_\mu \eta^{(j_2,k_2)} - g_4 A_\mu^{(j_2,k_2)}\right)\left(\frac{1}{v_4}\partial^\mu \eta^{(j_3,k_3)} - g_4 A^{(j_3,k_3)\,\mu}\right)\hat{\Phi}_0\,. \tag{6.27}
$$

Additional trilinear couplings of a spin-1 mode, coming from eq. (6.24), are given by

$$
\frac{i}{2} g_4\,\delta_{0,0,0}^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}\,\hat{\Phi}_0^\dagger \Big[\eta^{(j_1,k_1)}\,\left(\partial^\mu \eta^{(j_2,k_2)}\right) A_\mu^{(j_3,k_3)} - \left(\partial^\mu \eta^{(j_1,k_1)}\right) A_\mu^{(j_2,k_2)}\eta^{(j_3,k_3)}\Big]\,\hat{\Phi}_0\,, \tag{6.28}
$$

In the above two equations, one should express $\eta^{(j,k)}$ in terms of the mass eigenstates $\tilde{\eta}^{(j,k)}$ and $\tilde{A}_G^{(j,k)}$ by inverting eqs. (6.14):

$$
\eta^{(j,k)\,a} = \frac{1}{M_A^{(j,k)}}\left(M_{j,k}\tilde{\eta}^{(j,k)\,a} + g_4 v_4 \tilde{A}_G^{(j,k)\,a}\right)\,, \tag{6.29}
$$

except for the zero-mode, which is given by eq. (6.15). There are also trilinear interactions among scalars which include at least one $h^{(j,k)}$ field and no derivatives:

$$v_4\, h^{(j_1,k_1)} \left\{ -\frac{1}{2}\lambda_4\, \delta^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}_{0,0,0} h^{(j_2,k_2)} h^{(j_3,k_3)} + r_{j_2,k_2} r^*_{j_3,k_3} \delta^{(j_1,k_1)(j_2,k_2)(j_3,k_3)}_{0,3,1} \right.$$
$$\left. \times\, \hat{\Phi}^\dagger_0 \left(g_4 A_H^{(j_2,k_2)} + i\omega_{j_2,k_2} \tilde{\eta}^{(j_2,k_2)} \right) \left(g_4 A_H^{(j_3,k_3)} - i\omega_{j_3,k_3} \tilde{\eta}^{(j_3,k_3)} \right) \hat{\Phi}_0 \right\} , \quad (6.30)$$

where we defined the KK-number-dependent, dimensionless ratios

$$\omega_{j,k} = \frac{M_A^{(j,k)}}{v_4} \ . \quad (6.31)$$

The remaining trilinear couplings involve only $\tilde{\eta}^{(j,k)}$, $A_H^{(j,k)}$ and $A_G^{(j,k)}$ scalars, and can be derived from eq. (6.24).

The quartic interactions include terms involving $h^{(j,k)}$ and the spin-1 fields,

$$\frac{1}{2}\delta^{(j_1,k_1)\cdots(j_4,k_4)}_{0,0,0,0} h^{(j_1,k_1)} h^{(j_2,k_2)} \hat{\Phi}^\dagger_0 \left(\frac{1}{v_4}\partial_\mu \eta^{(j_3,k_3)} - g_4 A_\mu^{(j_3,k_3)} \right) \left(\frac{1}{v_4}\partial^\mu \eta^{(j_4,k_4)} - g_4 A^{(j_4,k_4)\mu} \right) \hat{\Phi}_0, \quad (6.32)$$

as well as the spinless fields $h^{(j,k)}$, $A_H^{(j,k)}$ and $\tilde{\eta}^{(j,k)}$:

$$h^{(j_1,k_1)} h^{(j_2,k_2)} \left\{ -\frac{1}{8}\lambda_4\, \delta^{(j_1,k_1)\cdots(j_4,k_4)}_{0,0,0,0} h^{(j_3,k_3)} h^{(j_4,k_4)} + r_{j_3,k_3} r^*_{j_4,k_4} \delta^{(j_1,k_1)\cdots(j_4,k_4)}_{0,0,3,1} \right.$$
$$\left. \times \hat{\Phi}^\dagger_0 \left(g_4 A_H^{(j_3,k_3)} + i\omega_{j_3,k_3} \tilde{\eta}^{(j_3,k_3)} \right) \left(g_4 A_H^{(j_4,k_4)} - i\omega_{j_4,k_4} \tilde{\eta}^{(j_4,k_4)} \right) \hat{\Phi}_0 \right\} . \quad (6.33)$$

The higher order terms in eqs. (6.24), (6.25) and (6.26) lead to additional quartic couplings among KK modes, which include at most a single 4D spin-1 field. These couplings are straightforward to derive but have long expressions so that we do not display all of them here.

We end this section by observing that the Abelian case can be recovered from the previous formulae by setting $\hat{\Phi}_0 = 1$, and considering a single gauge index a.

7. Conclusions

Gauge theories in six dimensions may be relevant for physics beyond the Standard Model provided the size of two dimensions is below 10^{-16} cm. Theories of this type have been proposed in the past, with compactification scales ranging from the electroweak scale to the GUT scale. This paper presents the first in-depth study of the gauge interactions among KK modes. We have concentrated on the simplest compactification of two dimensions that leads to zero mode fermions of definite 4D chirality, namely the "chiral square" defined in ref. [4].

After identifying a gauge fixing procedure appropriate for this compactification, we have determined a set of gauge invariant boundary conditions for gauge fields. The ensuing KK decomposition of a 6D gauge field A^α, $\alpha = 0, 1, \ldots, 5$, includes a tower of spin-1 modes,

$A_\mu^{(j,k)}$, that have a zero mode ($j = k = 0$), and two towers of spin-0 modes that have no zero mode, $A_4^{(j,k)}$ and $A_5^{(j,k)}$, where the pair of KK numbers (j, k) take the integer values $j \geq 1$, $k \geq 0$. The spin-1 zero mode is associated with the unbroken 4D gauge invariance, while the other spin-1 modes become heavy. Each nonzero spin-1 mode, $A_\mu^{(j,k)}$, has a longitudinal polarization given by a linear combination of $A_4^{(j,k)}$ and $A_5^{(j,k)}$. The linear combination $A_G^{(j,k)}$ of spin-0 modes that play the role of Nambu-Goldstone boson eaten by the heavy spin-1 mode depends on the KK numbers, as shown in eq. (2.21). The orthogonal combination, $A_H^{(j,k)}$, is gauge invariant and remains as an additional physical degree of freedom. Therefore, unlike the case of one extra dimension, where the extra components of the gauge field can be gauged away, any gauge field in two extra dimensions implies the existence of a tower of heavy spinless particles in the adjoint representation of the gauge group ("spinless adjoints"), whose interactions depend on the KK numbers.

The self-interactions of 6D non-Abelian gauge fields induce the following terms in the 4D Lagrangian involving KK modes: trilinear couplings of spin-1 modes, given in eq. (3.4); couplings of two spin-1 modes and one spin-0 mode, and couplings of three spin-0 modes, given in eq. (3.5); quartic couplings of spin-1 modes, couplings of two spin-1 modes and two spin-0 modes, and quartic couplings of spin-0 modes, given in eq. (3.9); and finally, couplings of one spin-1 mode, or one spinless adjoints, and two modes of the ghost field, given in eqs. (3.19) and (3.20), respectively.

The gauge interactions of a chiral 6D fermion induce couplings of a gauge field mode to fermion modes of both 4D chiralities. These depend on the 4D chirality of the zero-mode fermion. The spin-1 modes couple to fermion modes according to eqs. (4.8), (4.9), (4.11), and (4.12), while the Yukawa couplings of the spinless adjoints to the fermion modes are given in eqs. (4.10) and (4.13).

The gauge interactions of a 6D scalar field induce couplings of two scalar modes to one or two spin-1 modes, given in eq. (5.4) and (5.5), and also to one or two spinless adjoints, given in eqs. (5.6) and (5.7). A 6D quartic self-interaction of a scalar induces quartic couplings of the scalar modes, as in eq. (5.8).

We have also studied the case where the gauge symmetry is broken by a the VEV of a 6D scalar that has a zero mode. In this case, all gauge KK modes receive a contribution to their mass from the spontaneous breaking. The longitudinal polarizations of the heavy KK modes are now given by a linear combination of A_4, A_5 and the scalar that acquires the VEV, as shown in eq. (6.14). The longitudinal polarization of the would-be zero-mode gauge field is provided by the zero-mode of the additional spinless adjoint, as given in eq. (6.15). We showed how the longitudinal modes and additional scalars can be identified by studying the transformation properties under 6D gauge transformations. We displayed the interactions of these scalars with the spin-1 modes and among themselves in eqs. (6.27)–(6.33).

All the couplings among various KK modes mentioned above are induced at tree level by bulk interactions. Loop corrections generate operators localized at the three conical singularities of the chiral square. Even though these preserve KK parity, they lead to additional couplings among KK modes. These are perturbative corrections, but nevertheless may have important phenomenological consequences as they give rise to mixing among all KK modes belonging to the same tower. This effect is studied in a subsequent paper [14].

The detailed construction of 6D gauge theories with explicit boundary conditions presented here opens up various theoretical and model building avenues of research, including issues pertaining to symmetry breaking by boundary conditions [15], relation to other compactifications [16], the structure of gauge and gravitational anomalies on a compact space [17], and latticized or deconstructed [18] versions of the chiral square compactification.

Given the constraints on the compactification scale of universal extra dimensions from electroweak measurements [19], flavor-changing processes [20] or collider searches [11, 21], are as low as a few hundred GeV, it would be particularly interesting to use the tools developed here for analyzing the phenomenological implications of the Standard Model in two universal extra dimensions compactified on the chiral square. That model is well motivated by the natural happenstances of proton stability, constraint on the number of fermion generations, and existence of a dark matter candidate. More generally, the spectra and interactions derived here are relevant for any extensions of the Standard Model in the context of 6D theories.

Acknowledgments

We have benefited from the hospitality of the Aspen Center for Physics at various stages of this project. G.B. acknowledges the support of the State of São Paulo's Agency for the Promotion of Research (FAPESP), and the Brazilian National Council for Technological and Scientific Development (CNPq). The work of B.D. was supported by DOE under contract DE-FG02-92ER-40704.

References

[1] M.B. Green, J.H. Schwarz and P.C. West, *Anomaly free chiral theories in six-dimensions*, *Nucl. Phys.* **B 254** (1985) 327;
K.R. Dienes, E. Dudas and T. Gherghetta, *Grand unification at intermediate mass scales through extra dimensions*, *Nucl. Phys.* **B 537** (1999) 47 [hep-ph/9806292];
N. Arkani-Hamed, H.-C. Cheng, B.A. Dobrescu and L.J. Hall, *Self-breaking of the standard model gauge symmetry*, *Phys. Rev.* **D 62** (2000) 096006 [hep-ph/0006238];
M. Hashimoto, M. Tanabashi and K. Yamawaki, *Top mode standard model with extra dimensions*, *Phys. Rev.* **D 64** (2001) 056003 [hep-ph/0010260];
G.R. Dvali, S. Randjbar-Daemi and R. Tabbash, *The origin of spontaneous symmetry breaking in theories with large extra dimensions*, *Phys. Rev.* **D 65** (2002) 064021 [hep-ph/0102307];
J.M. Frere, M.V. Libanov and S.V. Troitsky, *Neutrino masses with a single generation in the bulk*, *JHEP* **11** (2001) 025 [hep-ph/0110045];
M. Fabbrichesi, M. Piai and G. Tasinato, *Axion and neutrino physics from anomaly cancellation*, *Phys. Rev.* **D 64** (2001) 116006 [hep-ph/0108039];
R.N. Mohapatra and A. Perez-Lorenzana, *Neutrino mass, proton decay and dark matter in TeV scale universal extra dimension models*, *Phys. Rev.* **D 67** (2003) 075015 [hep-ph/0212254];

T. Watari and T. Yanagida, *Higher dimensional supersymmetry as an origin of the three families for quarks and leptons*, *Phys. Lett.* **B 532** (2002) 252 [hep-ph/0201086];
T. Asaka, W. Buchmuller and L. Covi, *Bulk and brane anomalies in six dimensions*, *Nucl. Phys.* **B 648** (2003) 231 [hep-ph/0209144];
T. Asaka, W. Buchmuller and L. Covi, *Quarks and leptons between branes and bulk*, *Phys. Lett.* **B 563** (2003) 209 [hep-ph/0304142];
C. Biggio, F. Feruglio, I. Masina and M. Perez-Victoria, *Fermion generations, masses and mixing angles from extra dimensions*, *Nucl. Phys.* **B 677** (2004) 451 [hep-ph/0305129];
S. Gabriel, S. Nandi and G. Seidl, *6d higgsless standard model*, *Phys. Lett.* **B 603** (2004) 74 [hep-ph/0406020];
T. Ohl and C. Schwinn, *Unitarity, BRST symmetry and Ward identities in orbifold gauge theories*, *Phys. Rev.* **D 70** (2004) 045019 [hep-ph/0312263].

[2] B.A. Dobrescu and E. Poppitz, *Number of fermion generations derived from anomaly cancellation*, *Phys. Rev. Lett.* **87** (2001) 031801 [hep-ph/0102010];
N. Borghini, Y. Gouverneur and M.H.G. Tytgat, *Anomalies and fermion content of grand unified theories in extra dimensions*, *Phys. Rev.* **D 65** (2002) 025017 [hep-ph/0108094];
M. Fabbrichesi, R. Percacci, M. Piai and M. Serone, *Cancellation of global anomalies in spontaneously broken gauge theories*, *Phys. Rev.* **D 66** (2002) 105028 [hep-th/0207013].

[3] T. Appelquist, B.A. Dobrescu, E. Ponton and H.-U. Yee, *Neutrinos vis-a-vis the six-dimensional standard model*, *Phys. Rev.* **D 65** (2002) 105019 [hep-ph/0201131]; *Proton stability in six dimensions*, *Phys. Rev. Lett.* **87** (2001) 181802 [hep-ph/0107056].

[4] B.A. Dobrescu and E. Ponton, *Chiral compactification on a square*, *JHEP* **03** (2004) 071 [hep-th/0401032].

[5] E. Pontón and E. Poppitz, *Casimir energy and radius stabilization in five and six dimensional orbifolds*, *JHEP* **06** (2001) 019 [hep-ph/0105021].

[6] C. Csáki, C. Grojean and H. Murayama, *Standard model Higgs from higher dimensional gauge fields*, *Phys. Rev.* **D 67** (2003) 085012 [hep-ph/0210133].

[7] M. Hashimoto and D.K. Hong, *Topcolor breaking through boundary conditions*, *Phys. Rev.* **D 71** (2005) 056004 [hep-ph/0409223].

[8] M. Hashimoto, *Topcolor breaking through boundary conditions*, hep-ph/0502242.

[9] G. Servant and T.M.P. Tait, *Is the lightest Kaluza-Klein particle a viable dark matter candidate?*, *Nucl. Phys.* **B 650** (2003) 391 [hep-ph/0206071];
G. Servant and T.M.P. Tait, *Elastic scattering and direct detection of Kaluza-Klein dark matter*, *New J. Phys.* **4** (2002) 99 [hep-ph/0209262];
H.-C. Cheng, J.L. Feng and K.T. Matchev, *Kaluza-Klein dark matter*, *Phys. Rev. Lett.* **89** (2002) 211301 [hep-ph/0207125];
J.L. Feng, A. Rajaraman and F. Takayama, *Superweakly-interacting massive particles*, *Phys. Rev. Lett.* **91** (2003) 011302 [hep-ph/0302215]; *Graviton cosmology in universal extra dimensions*, *Phys. Rev.* **D 68** (2003) 085018 [hep-ph/0307375].

[10] H. Georgi, A.K. Grant and G. Hailu, *Brane couplings from bulk loops*, *Phys. Lett.* **B 506** (2001) 207 [hep-ph/0012379].

[11] H.-C. Cheng, K.T. Matchev and M. Schmaltz, *Radiative corrections to Kaluza-Klein masses*, *Phys. Rev.* **D 66** (2002) 036005 [hep-ph/0204342].

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[12] H.-C. Cheng, K.T. Matchev and M. Schmaltz, *Bosonic supersymmetry? getting fooled at the LHC*, *Phys. Rev.* **D 66** (2002) 056006 [hep-ph/0205314].

[13] M. Carena, T.M.P. Tait and C.E.M. Wagner, *Branes and orbifolds are opaque*, *Acta Phys. Polon.* **B33** (2002) 2355 [hep-ph/0207056].

[14] G. Burdman, B.A. Dobrescu and E. Pontón, in preparation.

[15] A. Hebecker and J. March-Russell, *The structure of GUT breaking by orbifolding*, *Nucl. Phys.* **B 625** (2002) 128 [hep-ph/0107039];
L.J. Hall, Y. Nomura, T. Okui and D.R. Smith, SO(10) *unified theories in six dimensions*, *Phys. Rev.* **D 65** (2002) 035008 [hep-ph/0108071].

[16] C.A. Scrucca, M. Serone, L. Silvestrini and A. Wulzer, *Gauge-higgs unification in orbifold models*, *JHEP* **02** (2004) 049 [hep-th/0312267];
A. Hebecker and M. Ratz, *Group-theoretical aspects of orbifold and conifold guts*, *Nucl. Phys.* **B 670** (2003) 3 [hep-ph/0306049];
A. Hebecker, *Grand unification in the projective plane*, *JHEP* **01** (2004) 047 [hep-ph/0309313].

[17] T. Gherghetta and A. Kehagias, *Anomaly cancellation in seven-dimensional supergravity with a boundary*, *Phys. Rev.* **D 68** (2003) 065019 [hep-th/0212060];
K.R. Dienes, *Shape versus volume: making large flat extra dimensions invisible*, *Phys. Rev. Lett.* **88** (2002) 011601 [hep-ph/0108115];
K.R. Dienes and A. Mafi, *Shadows of the planck scale: the changing face of compactification geometry*, *Phys. Rev. Lett.* **88** (2002) 111602 [hep-th/0111264];
K.R. Dienes and A. Mafi, *Kaluza-Klein states versus winding states: can both be above the string scale?*, *Phys. Rev. Lett.* **89** (2002) 171602 [hep-ph/0207009].

[18] C.T. Hill, S. Pokorski and J. Wang, *Gauge invariant effective lagrangian for Kaluza-Klein modes*, *Phys. Rev.* **D 64** (2001) 105005 [hep-th/0104035];
N. Arkani-Hamed, A.G. Cohen and H. Georgi, *(De)constructing dimensions*, *Phys. Rev. Lett.* **86** (2001) 4757 [hep-th/0104005];
H.-C. Cheng, C.T. Hill, S. Pokorski and J. Wang, *The standard model in the latticized bulk*, *Phys. Rev.* **D 64** (2001) 065007 [hep-th/0104179].

[19] T. Appelquist, H.-C. Cheng and B.A. Dobrescu, *Bounds on universal extra dimensions*, *Phys. Rev.* **D 64** (2001) 035002 [hep-ph/0012100];
T. Appelquist and H.-U. Yee, *Universal extra dimensions and the Higgs boson mass*, *Phys. Rev.* **D 67** (2003) 055002 [hep-ph/0211023];
J.F. Oliver, J. Papavassiliou and A. Santamaria, *Universal extra dimensions and* $Z \to b\bar{b}$, *Phys. Rev.* **D 67** (2003) 056002 [hep-ph/0212391].

[20] K. Agashe, N.G. Deshpande and G.H. Wu, *Universal extra dimensions and* $b \to s\gamma$, *Phys. Lett.* **B 514** (2001) 309 [hep-ph/0105084];
T. Appelquist and B.A. Dobrescu, *Universal extra dimensions and the muon magnetic moment*, *Phys. Lett.* **B 516** (2001) 85 [hep-ph/0106140];
D. Chakraverty, K. Huitu and A. Kundu, *Effects of universal extra dimensions on* $B^0\bar{B}^0$ *mixing*, *Phys. Lett.* **B 558** (2003) 173 [hep-ph/0212047];
A.J. Buras, M. Spranger and A. Weiler, *The impact of universal extra dimensions on the unitarity triangle and rare K and B decays*, *Nucl. Phys.* **B 660** (2003) 225 [hep-ph/0212143];
A.J. Buras, A. Poschenrieder, M. Spranger and A. Weiler, *The impact of universal extra dimensions on* $B \to X_s\gamma$, $B \to X_s gluon$, $B \to X_s\mu^+\mu^-$, $K_L \to \pi^0 e^+e^-$ *and* ϵ'/ϵ, *Nucl. Phys.* **B 678** (2004) 455 [hep-ph/0306158].

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[21] T.G. Rizzo, *Probes of universal extra dimensions at colliders*, *Phys. Rev.* **D 64** (2001) 095010 [hep-ph/0106336];
C. Macesanu, C.D. McMullen and S. Nandi, *Collider implications of universal extra dimensions*, *Phys. Rev.* **D 66** (2002) 015009 [hep-ph/0201300]; *New signal for universal extra dimensions*, *Phys. Lett.* **B 546** (2002) 253 [hep-ph/0207269];
F.J. Petriello, *Kaluza-Klein effects on Higgs physics in universal extra dimensions*, *JHEP* **05** (2002) 003 [hep-ph/0204067];
C.D. Carone, J.M. Conroy, M. Sher and I. Turan, *Universal extra dimensions and Kaluza-Klein bound states*, *Phys. Rev.* **D 69** (2004) 074018 [hep-ph/0312055];
C. Macesanu, S. Nandi and M. Rujoiu, *Single Kaluza-Klein production in universal extra dimensions*, *Phys. Rev.* **D 71** (2005) 036003 [hep-ph/0407253];
M. Battaglia, A. Datta, A. De Roeck, K. Kong and K.T. Matchev, *Contrasting supersymmetry and universal extra dimensions at the clic multi-TeV* e^+e^- *collider*, *JHEP* **07** (2005) 033 [hep-ph/0502041].